Belief Propagation and Potts Model

Tomasz M. Gwizdałła

University of Łódź, Dept. of Solid State Physics
Pomorska 149/153, 90-236 Łódź, Poland
email: tomgwizd@uni.lodz.pl

Abstract

In the paper there are presented the results of steady state searching for 3-5 state Potts model. The results of simulation are presented and its efficiency is discussed. Finally the model is adapted to the trial of reconstruction of grey scaled images initially blurred by the special modification procedure. The results show the usefulness of the simple method in the process of contour recognition.

1 Introduction

The idea of Loopy Belief Propagation algorithm is an interesting approach which encompasses in its theoretical and application formulation problems of physical as well as image analysis problems. In the still arising number of current studies let us especially mention those papers which clearly shown these connections presenting the questions of BP in th language of various parts of physics. Chertkov and all [1] studied the solvability of inference model on graph using Kasteleyn model. The problem of graph partitioning, connected to the sociophysical issue of community detection was studied by the means of BP [5]. An interesting enhancement with the notion of superpixel adapted to the dense graphs was proposed by Ngowu [6]. Another works were directly devoted to the problems of image reconstruction: Yin[10] looked for a moving objects in a 3D mesh, Guan studied robot vision using Sobel edge filter[3]. Special attention can be put on works of Felzenszwalb [2] who proposed few operators for early vision problem and Tanaka [8] who analysed 4-state Potts model based BP on the randomly modified image using thermodynamical approach.

In this paper we are going to present some remarks concerning the possibility of study of n-state Potts model using Belief Propagation and how it can be related to some specific images reconstruction.

2 Standard and vector Potts model

The idea to use the method called Belief Propagation to the study of magnetic propertie s of solid systems has arisen since the papers of Yedidia and coworkers [9] and has been in some preliminary version presented in our previous paper [4]. Let us now shortly present some main ideas which would be helpful in further reading.

One has to start with the construction of a system of nodes for which the specific total state can be described by the probability related to some multi-state cluster probabilities.

These are obtained self-consistently by the exchanging of messages between nodes. Here especially two factors have to be taken into consideration. The first one is the form of weights determining the influence of external field as well as node-node interaction .. Usually one uses the formulas of exponential character:

$$\Phi_i(x_i) = exp(hs_i), \Psi_{ij}(x_i, x_j) = exp\Big(\frac{Js_i s_j}{T}\Big) \tag{1}$$

where i and j enumerates neighbouring spins, h is th external field and J the exchange integral or coupling constant and describes the interaction between nodes being in states s_i and s_j. Such an attempt is however necessary only when we deal with the problem which has unambiguous physical formulation and we have to determine such physical quantities like internal energy, entropy or finally free energy. In other applications another formulas can be used, for comparison see e.g. [2].

The second element is the complexity of message creation. It is connected to the order of approximation measured in this approach with the size of clusters exchanging the information. In the most simple model (BP) the message passed between nodes is created on the basis of only nearest neighbours. For the more sophisticated approaches one has to build larger clusters and specify in detail the relations between different nodes belonging to different or overlapping clusters. Certainly the increase of order of approximation which can be connected to the successive steps of Kikuchi model gives still better results for physically interpretable values what has been shown in our previous paper [4] for the BP and GeneralBP approximation of well known Ising model.

In the paper we deal with the generalization of Ising model, so called Potts model [7]. The major change introduced into this model as compared to the Ising one is the enhancement of the number of spin states to the arbitrarily high value. In this sense one can say that Potts model has two boundary cases: the decrease of the number of states leads to the Ising model while its increase to infinity produces the XY model which is the classical version of the Heisenberg model. Practically one can say, that exceeding the number of state be equal to about 10 we are not able to observe the changes of different critical parameters, like phase transition temperature.

The notion of Potts model isn't unambiguous itself because its definition vas been evolving since formulation. Let us denote the states of neighbouring nodes by $k, l \in 0, 1, ..., n_{states} - 1$. In the below equations ΔH means only the contribution to the total energy coming from the selected pair.

Initially Potts used the form of interaction presented by equation 2. This form is usually called planar or vector. Now, more often so called standard form (3) is in use. For some reasons which will be explained later we define here a new approach (we will call it modified 2), based on the vector one which has indeed less physical sense. It will be however useful in some further studies.

$$\Delta H = -Jcos\Big(\frac{2\pi(k-l)}{n_{states}}\Big) \tag{2}$$

$$\Delta H = -J\delta_{kl} \tag{3}$$

$$\Delta H = -Jcos\Big(\frac{\pi(k-l)}{n_{states}}\Big) \tag{4}$$

One should also point the possible differences in the sign of J value. The positive exchange constant corresponds to the ferromagnetic ordering at low temperatures while negative to the antiferromagnetic one where neighbouring spins are ordered antiparallely.

Let us now shortly sketch some main features of results obtained for Potts model and their relation to the properties of further image recognition process. It is well known that the steady state estimation in the BP-like algorithms is the hardest task for the temperatures close to the so called phase transition point. It can be defined as a point where the internal order of sample, in the Belief Propagation model expressed by a set of beliefs, visualized by the spin direction scheme and defined by the J exchange constant changes the form. The best known phase transition related to the magnetic problems is the ferromagnetic (spontaneous parallel magnetisation of all spins in sample) to paramagnetic (total value of magnetization equal to 0 without any ordering in substructures) transition observed e.g. for Ising model. [4]. For the antiferromagnetic initial ordering the final (high temperature) state looks same but in low temperatures there are the forms of chains created in the sample with the successive spins oriented atiparallely.

The most simple versions for 3 to 5 states were studied in order to determine main physical values as well as information about the efficiency of calculations. For both classical formulations one can observe distinct phase transition when going to the larger temperatures. J is here certainly positive and for vector model the critical temperature decreases while for standard one is constant. For the modified model the "magnetization" doesn't reach zero value and this fact follows clearly the interpretation of magnetization vector for single site. In the Potts model it is assumed that the state k means that the angle between the selected zero angle and a vector is equal to $2k\pi/(n_{states} - 1)$ so the equal belief of finding the vector in any possible state means averaging the equal vectors distributed uniformly in the interval of full angle. For the modified model this condition isn't certainly satisfied. On the plot we present the results for both ferromagnetic (starting from 1) and antiferromagnetic coupling.

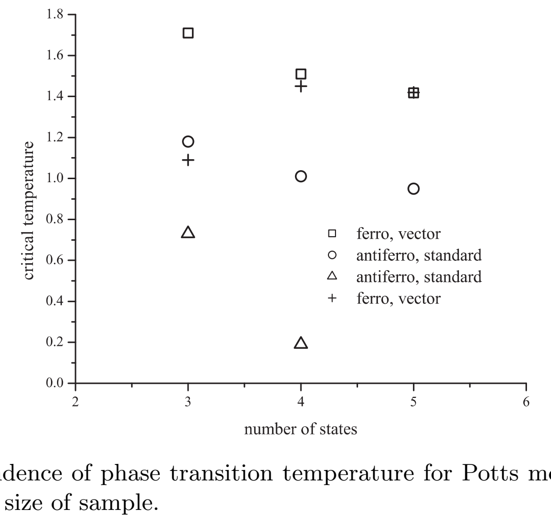

Figure 1. The dependence of phase transition temperature for Potts model (calculated using BP algorithm) on the size of sample.

The influence of number of states on the phase transition temperature is presented on figure 1. It may be noticed that contrary to the ferro-paramagnetic transition point which decreases with increasing number of states for the antiferro initial ordering we cannot formulate one conclusion. The transition point can can oscillate as well as decrease to the complete lack of it. The major information following the presented plot is about the region of main interest during further part of work. As it will be explained later the region of phase transition is the most probable temperature interval for the recognition process. Using the classical attempts one can expect the best results for the temperatures up to 1.6, while, as it can be seen from fig.2, for the modfied one it can be more than 2.0 (the curvature leading to stabilization).

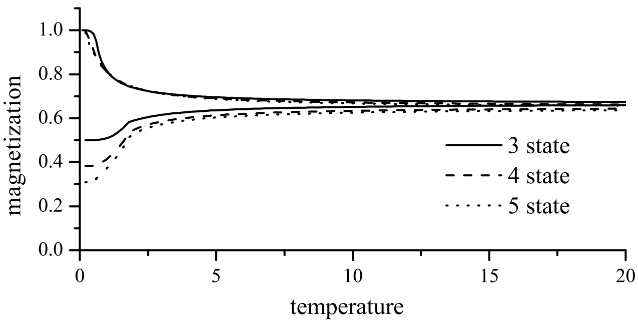

Figure 2. The temperature dependence of magnetization for modified Potts models.

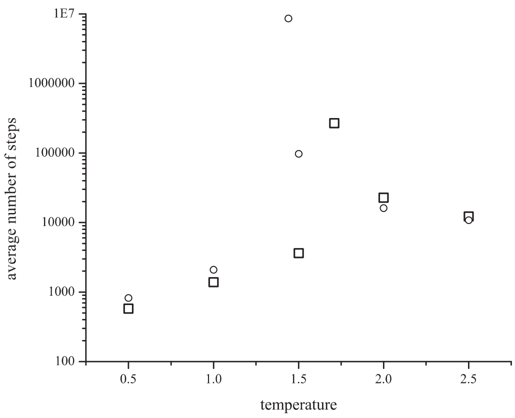

Figure 3. The efficiency of obtaining the steady state.

What is especially important from the point of view of further calculations is the efficiency of determining the magnetization. On the fig.3 we have presented the number of steps required to obtain the steady state of Potts sample for different temperatures and 3 (squares) and 4 (circles) possible states. We assumed that we reach a steady state when we know correctly 6 digits of the result of calculation. The points are spread over temperature scale and we marked out also those temperatures where the phase transition occur. Please notice the logaritmic scale on vertical axis and huge increase of necessary steps number for critical conditions.

3 2D image recognition

As it was written earlier there exist some approaches using BP technique in image recognition. It is interesting that all of them use the ferromagnetic coupling. Let us now to show some result obtained with negative J value. We want to check few features: are we able to use such a relatively simple technique in image analysis; for which temperature the correct recognition occur, is this temperature related to the phase transition point, what would be the efficiency of recognition in the context of picture 3. Because the result of calculations performed using BP model isn't characterized with the size effect we decided to start from the small sizes of images 10x10 and 20x20. All initial images are filled with one of four values corresponding to the one of the four points on the grey scale (from white to black). Then the blurring procedure were done. We use the cellular automata like algorithm where the new value of the selected point is determined as a combination of its initial value and the averaged and weighted (the weight may be understand as the blurring parameter) value of nearest neighbours. Such a modified picture was used as a source of local value of external field in formula 1.

One can generally expect some properties of this method. the temperature in formula 1 is indeed responsible for the change of relative significance of constant and interaction based factor in message determination. Therefore for low temperatures when the interaction between spins/states determines the character of a sample the result should reflect the antiferromagnetic coupling. For the extremely high temperatures the constant factor should dominate and the obtained picture should be the same as a source one.

On the fig. 4 there are shown some of patterns used in the procedure. Four upper have the edges 10x10 while two lower 20x20. The left column always contains the original image, in the middle there are modified using blurring algorithm and the in the right column there is a reconstructed picture. The successive pictures can be described as follows: (1) "A" letter slightly blurred, (2) "A" letter strongly blurred, (3) island with different intensities, (4) completely random, (5) "A" letter strongly blurred, (6) stripes. First four configurations were selected in order to determine some general features of algorithm, building the fifth we want to check whether the procedure will be succesful for the greater sample of the same type and the sixth was a response to the fact that procedure occurred to be efficient in the contour recognition.

We can also try to describe the efficiency of the algorithm using two functions to determine the distance between the obtained result and the original or blurred picture.

$$
\begin{aligned}
f_1(img_1, img_2) &= \sum_{i,j} \left(s_{ij}^{(1)} - s_{ij}^{(2)} \right)^2 \qquad (5) \\
f_2(img_1, img_2) &= \sum_{i,j} |s_{ij}^{(1)} - s_{ij}^{(2)}|
\end{aligned}
$$

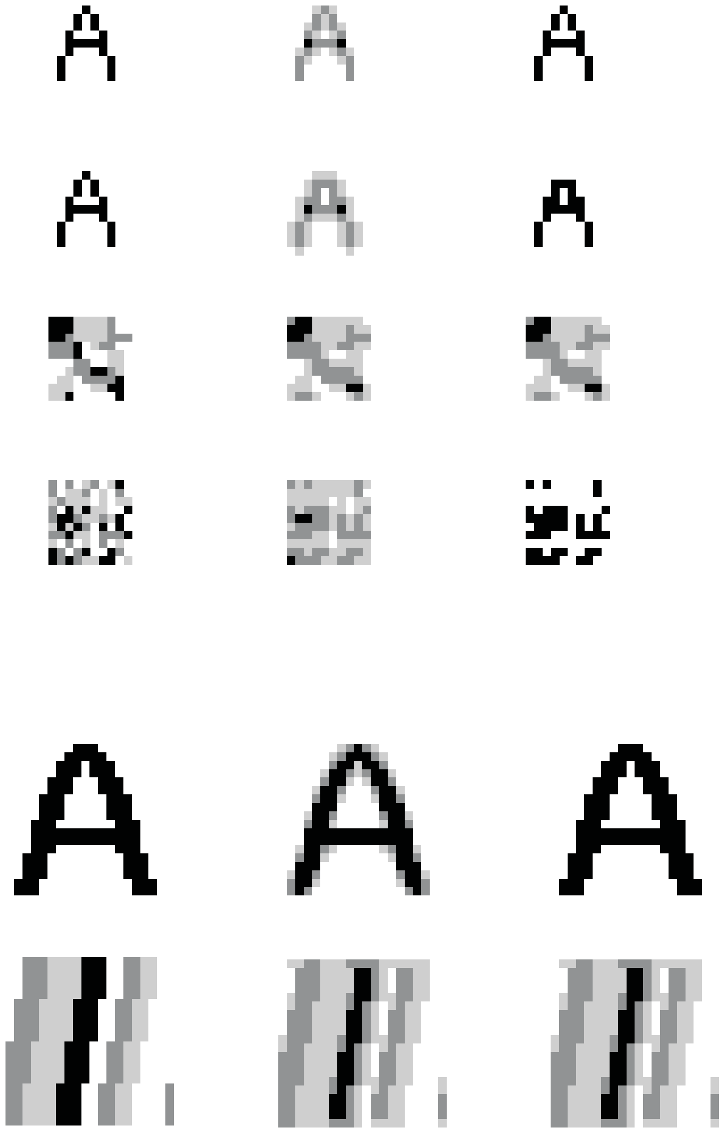

Figure 4. Patterns used in recognition process.

Te second (linear) formula is similar to the extended Hamming distance popular in different algorithmical solutions while the square one, popular rather in physical problems, better signs the bigger single deviations.

The plot of difference calculated with formula 6 for three images(little "A"(1), big "A" (3), random (4)) is presented on fig.5. For the well reconstructed pictures there exist quite wide plateau where the difference is equal to zero. Even for badly reconstructed

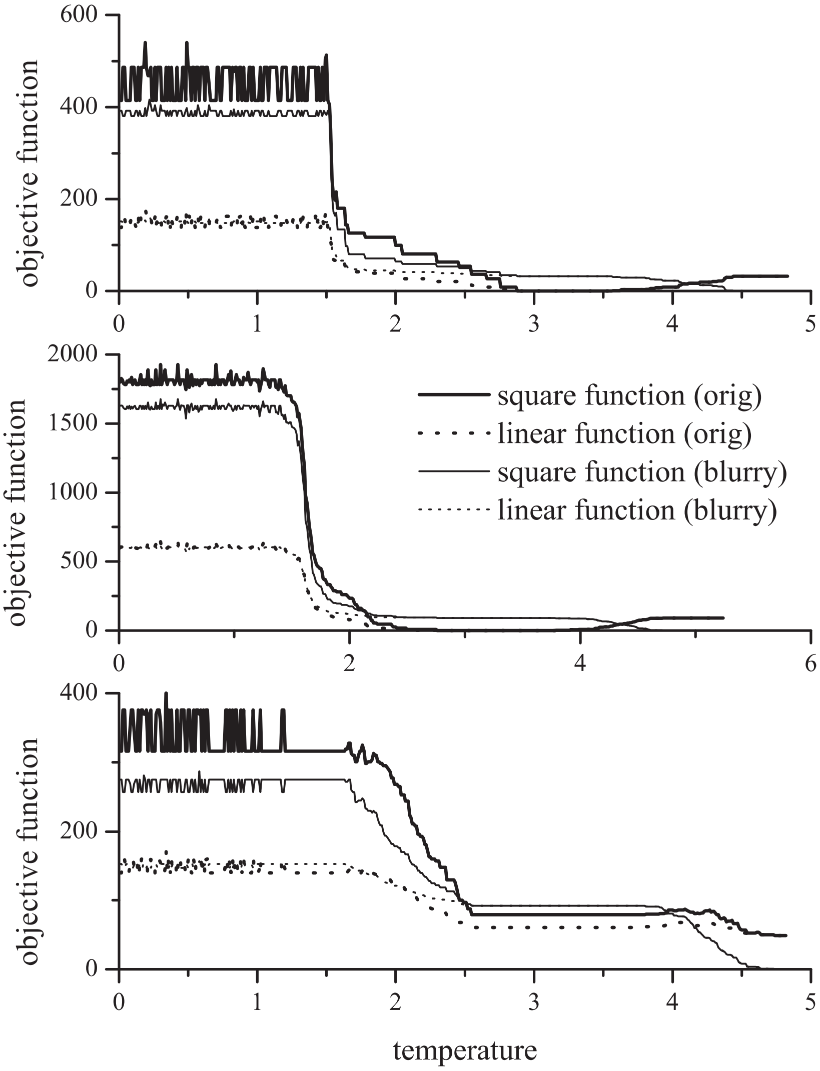

Figure 5. Efficiency of recognition process.

patterns it exist a region of temperatures where the picture is more similar to the original one than to the blurred one which accordingly to the earlier remarks dominates in the final part of plot. It is interesting that although the intervals of temperatures of good recognition for different sizes overlap together they are generally different. One should also point out that a number of steps needed to solve such a problem is very low, it doesn't exceed 10000.

4 Conclusions

The use of Loopy Belief Propagation for two-dimensional image reconstruction may produce correct results with high efficiency for some specific cases. When take into account the antiferromagnetic coupling constant the procedure can recognize especially the contours of objects sharpening it with two more distant, in the sense of model, colors. One has to point out few features of presented algorithm. First of all one cannot use the pure two-state Ising model due to more complicated initial state. The second problem could arise when one would try to represent with Potts-like model pictures with more number of colours enhancing presented here grey-scale approach.

Bibliography

[1] Michael Chertkov, Vladimir Y Chernyak, and Razvan Teodorescu. Belief propagation and loop series on planar graphs. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(05):P05003 (19pp), 2008.

[2] P.F. Felzenszwalb and D.P. Huttenlocher. Efficient Belief Propagation for early vision. *International Journal of Computer Vision*, 70:41–53, 2006.

[3] S. Guan and R. Klette. Belief-propagation on edge images for stereo analysis of image sequences. In *Lecture Notes In Computer Science; Vol. 4931/2008, Robot Vision*, pages 291–302. Springer-Verlag, 2008.

[4] T.M. Gwizdałła. The critical properties of magnetic system using Generalized Belief Propagation technique. In *Proceedings of XI KAEiOG Conference, Szymbark*, pages 85–94, 2008.

[5] M. B. Hastings. Community detection as an inference problem. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, 74(3):035102, 2006.

[6] I. Nwogu and J.J. Corso. (BP)2: Beyond pairwise Belief Propagation labeling by approximating Kikuchi free energies. In *IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, pages 1–8, 2008.

[7] R.B. Potts. Some generalized order-disorder transformations. *Proc.Cambr.Phil.Soc.*, 48:106–109, 1952.

[8] Kazuyuki Tanaka and D. M. Titterington. First-order phase transition and Bayesian Image Processing by Loopy Belief Propagation. *Progress of Theoretical Physics Supplement*, 157:288–291, 2005.

[9] J.S. Yedidia, W.T. Freeman, and Y Weiss. Bethe free energy, Kikuchi approximations and belief propagation algorithms. In *MERL Report*, 2000.

[10] Zhaozheng Yin and Robert Collins. Belief Propagation in a 3D Spatio-temporal MRF for Moving Object Detection. *Computer Vision and Pattern Recognition, IEEE Computer Society Conference on*, 0:1–8, 2007.