Can interval computations be applied over spaces of non-numbers ?

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Abstract Interval methods proved to be a useful tool for solving global optimization and nonlinear equations systems problems over $\mathbb{R}^n$. But an interval may be defined not only over the set of real numbers or real vectors, but over any partially ordered set. The paper shows how basic ideas of interval computations can be generalized for such spaces. Some specific applications are proposed and preliminary computational results are presented.

1 Introduction

By an interval one usually means the set of real numbers: $[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$. It is however not the only meaning of the word "interval". According e.g. to Wikipedia ([12]), we may extend the above definition to intervals over any partially ordered sets, replacing $\mathbb{R}$ by a partially ordered set $(P, \le)$, where "$\le$" is any p.o. relation.

In fact, authors from the interval community call vectors (or matrices) of intervals: "vector intervals" ("matrix intervals") as well as "interval vectors" (or "interval matrices"), i.e. they can be described either as vectors (matrices) of interval elements or as intervals over the set of real vectors (matrices). Moreover, the term "vector interval" is actually more accurate than "interval vector" – the set of intervals is not a vector space.

This paper tries to extend the idea of interval computations to intervals over any poset (i.e. partially ordered set), not only over $\mathbb{R}$ ($\mathbb{R}^n$ or even $\mathbb{R}^{n\times m}$).

2 Basic assumptions

Consider the triple $\Big(\mathbb{X},\, d(\cdot,\cdot),\, \preceq\Big)$, where $(\mathbb{X}, d)$ is a metric space and "$\preceq$" is a partial order on $\mathbb{X}$. The metric d and the partial order "$\preceq$" should fulfill the following axiom – let us call it the metric–order axiom (MO):

$$x_1, x_2, x_3 \in \mathbb{X} \text{ and } x_1 \preceq x_2 \preceq x_3 \;\Rightarrow\; d(x_1, x_2) \le d(x_1, x_3) \ . \qquad \text{(MO)}$$

Definition 2.1. A (closed) *interval* $[\underline{x}, \overline{x}]$ of elements of the space $\mathbb{X}$ is the set $\{x \in \mathbb{X} \mid \underline{x} \preceq x \preceq \overline{x}\}$, where $\underline{x}, \overline{x} \in \mathbb{X}$ and $\underline{x} \preceq \overline{x}$.

Consistently with [6], we denote the space of intervals over the set $\mathbb{X}$ by $\mathbb{IX}$, e.g. $\mathbb{IR}$, $\mathbb{IC}$, $\mathbb{IR}^n$. Open intervals, i.e. intervals excluding one or both of its endpoints are denoted as $]\underline{x}, \overline{x}]$, $[\underline{x}, \overline{x}[$, $]\underline{x}, \overline{x}[$.

Several authors (e.g. [3], [5], [9], [10], [11]) generalize the notion of an interval and use so-called "improper intervals", where $\underline{x} \succeq \overline{x}$. Such intervals are (usually) interpreted as intervals with changed quantifier ("$\forall$" and "$\exists$") in expressions, where they are used. The set of all Kaucher intervals (i.e. where $\underline{x} \preceq \overline{x}$ does not have to be fulfilled) over the space $\mathbb{X}$ will be denoted by $\mathbb{KX}$, which is compatible with [6], too.

Remark 2.2. Consider the space $\mathbb{IX}$ of intervals. The space of n-component interval vectors is equivalent to the space of intervals over $\mathbb{X}^n$, with the partial order "$\preceq$" for $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ defined by $x \preceq y$, iff $x_i \preceq y_i \quad \forall i = 1, \ldots, n$. Also the metric can be obtained in one of a few obvious ways (l_1, l_2, l_∞).

Hence, there is no reason to distinguish between $\mathbb{I}(\mathbb{X}^n)$ and $(\mathbb{IX})^n$, we simply use the notation $\mathbb{IX}^n$.

Remark 2.3. We identify the set $\mathbb{X}$ with the set of degenerate intervals over $\mathbb{X}$, i.e. intervals, for which $\underline{x} = \overline{x}$. So, e.g. $\big(\mathbb{IX} \setminus \mathbb{X}\big)$ is the set of all non-degenerate intervals.

3 Bisection

The bisection – main tool of all branch-and-bound algorithms – subdivides an interval into two intervals such that their union is equal to the original one.

3.1 Definition

We shall treat bisection as a function of two arguments: an interval $\boldsymbol{x}$ and another argument $k \in K(\boldsymbol{x})$, describing the mode of bisection. The set $K(\boldsymbol{x})$ depends on the properties of $\mathbb{X}$ and on the interval $\boldsymbol{x} \in \mathbb{IX}$. In particular it may be a trivial set $K(\boldsymbol{x}) = \{e\}$, if only $\boldsymbol{x}$ is non-degenerate; this is the case e.g. for (one-dimensional) real intervals.

We shall define bisection only for non-degenerate intervals; it is not clear for the author if there is a need for a convention on bisecting a single point. It seems we can assume $K\big([\underline{x}, \overline{x}]\big) = \emptyset$, in the case when $\underline{x} = \overline{x}$, for consistency.

So, bisection is a function: $b\colon \{(\boldsymbol{x}, k) \mid \boldsymbol{x} \in \mathbb{IX} \setminus \mathbb{X}, k \in K(\boldsymbol{x})\} \to \mathbb{IX} \times \mathbb{IX}$, that fulfills certain conditions. We claim it should fulfill the following axioms:

$$b\big([\underline{x}, \overline{x}], k\big) = \Big([\underline{x}, \overline{m}],\, [\underline{m}, \overline{x}]\Big),\ \underline{m}, \overline{m} \in \mathbb{X}\ , \tag{B1}$$

$$[\underline{x}, \overline{m}] \cup [\underline{m}, \overline{x}] = [\underline{x}, \overline{x}]\ . \tag{B2}$$

$$\underline{m} \neq \underline{x} \text{ and } \overline{m} \neq \overline{x} \tag{B3}$$

Comments. Axiom (B3) simply means that none of the resulting intervals is equal to the bisected one. In axiom (B2) the equality may possibly be relaxed to "$\supseteq$".

The following properties are implied by above axioms.

Property 3.1.1. Let $b\big([\underline{x}, \overline{x}], k\big) = \Big([\underline{x}, \overline{m}],\, [\underline{m}, \overline{x}]\Big)$. Then $\underline{m} \in]\underline{x}, \overline{x}]$ and $\overline{m} \in [\underline{x}, \overline{x}[$.

The proof comes directly from (B2) and (B3).

For dense spaces also more important properties may be obtained.

Property 3.1.2. Let $b\big([\underline{x},\overline{x}],k\big) = \Big([\underline{x},\overline{m}],\ [\underline{m},\overline{x}]\Big)$ and $(\mathbb{X},\preceq)$ be dense. Then:

$$\underline{m} \neq \overline{x} \text{ and } \overline{m} \neq \underline{x}\ , \tag{1}$$

Proof. Suppose $\underline{m} = \overline{x}$. Then the interval $[\underline{m},\overline{x}]$ contains only one point and the interval $[\underline{x},\overline{m}]$ has to contain all other points from $[\underline{x},\overline{x}]$. But hence $\boldsymbol{X}$ is a dense set, there are points from $[\underline{x},\overline{x}]$ that may be arbitrary close to $\overline{x}$. So, the only possible value of $\overline{m}$ is $\overline{x}$. But this contradicts (B2).

Analogously, we can prove that $\overline{m} = \underline{x}$ contradicts (B2). QED.

Please note that for $(\mathbb{X},\preceq)$ that is not dense (1) does not have to be fulfilled. For example for the integer interval $[1..2]$ we have – according to definitions in the next subsection – $b\big([1..2],k\big) = \Big([1..1],[2..2]\Big)$.

Property 3.1.3. Let $b\big([\underline{x},\overline{x}],k\big) = \Big([\underline{x},\overline{m}],\ [\underline{m},\overline{x}]\Big)$, $(\mathbb{X},\preceq)$ be dense. Then $\overline{m} \not\preceq \underline{m}$.

Proof. If $\overline{m} \preceq \underline{m}$ then interval $[\overline{m},\underline{m}]$ is nonempty and contained in $[\underline{x},\overline{x}]$, but in none of the resulting intervals $[\underline{x},\overline{m}]$ and $[\underline{m},\overline{x}]$. This contradicts (B2). QED.

Again, for non-dense spaces it does not have to hold, e.g. the above bisection of integer intervals.

So, for dense spaces, $\overline{m} \not\preceq \underline{m}$. But can it be strengthened to $\underline{m} \preceq \overline{m}$? It seems only an under an additional condition.

Property 3.1.4. Let $b\big([\underline{x},\overline{x}],k\big) = \Big([\underline{x},\overline{m}],\ [\underline{m},\overline{x}]\Big)$, $(\mathbb{X},\preceq)$ be dense and the following condition holds: $\forall x \in \mathbb{X}$ the set:

$$N(x) = \{y \in \mathbb{X} \mid y \not\preceq x \text{ and } x \not\preceq y\} \tag{2}$$

is an open set in the topology generated by metric d. Then: $\underline{m} \preceq \overline{m}$.

Proof. Suppose $\underline{m} \not\preceq \overline{m}$.

Hence $\underline{m} \in N(\overline{m})$ – from (2) – and hence $N(\overline{m})$ is an open set, there is an open neighborhood U of $\underline{m}$ such that $\forall x \in U\ x \not\preceq \overline{m}$. Since $\underline{m} \in]\underline{x},\overline{x}[$, we can choose $U \subseteq]\underline{x},\overline{x}[$. And hence $(\mathbb{X},\preceq)$ is dense, there is an element $m' \in U$ such that $\underline{m} \preceq m'$ and – hence $\underline{m} \not\preceq \overline{m}$ – $m' \not\preceq \overline{m}$.

So $m' \notin [\underline{x},\overline{m}]$ and $m' \notin [\underline{m},\overline{x}]$. But $m' \in U \subseteq]\underline{x},\overline{x}[$, which contradicts (B2). QED.

Comment. All continuous spaces $\mathbb{X}$ considered by the author fulfilled the condition of openness of (2). Nevertheless it does not seem to be implied by earlier assumptions in any way. The author is uncertain about the existence (and usefulness) of spaces, where (2) does not have to be open and intervals over such spaces.

3.2 Examples

Intervals from $\mathbb{IR}$. In this case the most commonly used bisection operation subdivides the interval $[\underline{x},\overline{x}]$ to $[\underline{x},c]$ and $[c,\overline{x}]$, where $c = \mathrm{mid}\,\boldsymbol{x} = \frac{1}{2}\cdot(\underline{x}+\overline{x})$. The set $K(\boldsymbol{x})$ contains only one element (if $\boldsymbol{x}$ is non-degenerate). Such an operation obviously fulfills all the bisection axioms.

Intervals from $\mathbb{IR}^n$. Now, we can do the bisection in n ways, bisecting one of the n coordinates. So,$K(\boldsymbol{x}) = \{k \in \{1, 2, \ldots, n\} \mid \boldsymbol{x}_k$ is non-degenerate $\}$. We choose $k \in K(\boldsymbol{x})$ and bisect $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)^T$ to $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$, where:

$$\begin{aligned} \underline{x}^{(1)} &= (\underline{x}_1, \ldots, \underline{x}_n)^T , & \overline{x}^{(1)} &= (\overline{x}_1, \ldots, \overline{x}_{k-1}, c, \overline{x}_{k+1}, \ldots, \overline{x}_n)^T , \\ \overline{x}^{(2)} &= (\overline{x}_1, \ldots, \overline{x}_n)^T , & \underline{x}^{(2)} &= (\underline{x}_1, \ldots, \underline{x}_{k-1}, c, \underline{x}_{k+1}, \ldots, \underline{x}_n)^T , \end{aligned}$$

and $c = \mathrm{m}id\, \boldsymbol{x}_k$, as previously. Axioms of bisection are obviously fulfilled with $\underline{m} = \underline{x}^{(2)}$ and $\overline{m} = \overline{x}^{(1)}$.

Integer intervals. Following [2], we denote integer intervals by $[n_1..n_2]$, where $n_1 \leq n_2$. Obviously, $[n_1..n_2] = \{n_1, n_1 + 1, n_1 + 2, \ldots, n_2 - 1, n_2\}$.

$K\big([\underline{n}..\overline{n}]\big) = \{e\}$ (when $\underline{n} + 1 \leq \overline{n}$) is trivial, as for reals and the bisection is given by the following formula: $b\big([n_1..n_2], k\big) = \Big([n_1..m], [m + 1..n_2]\Big)$, where $m = \Big\lfloor \frac{n_1+n_2}{2} \Big\rfloor$ and $\lfloor \cdot \rfloor$ is the integer part of the number .

Some more interesting examples are going to be considered below.

3.3 Problem one

Please note that the above conditions do not assure that the bisection will in general be convergent to a point, i.e. that a sequence:

$$\begin{aligned} &(\boldsymbol{x}_n)_{n \in \mathbb{N}} \text{ , such that:} \\ &b(\boldsymbol{x}_n, k_n) = \big(\boldsymbol{x}_{n+1}, \boldsymbol{z}_{n+1}\big) \text{ or } b(\boldsymbol{x}_n, k_n) = \big(\boldsymbol{z}_{n+1}, \boldsymbol{x}_{n+1}\big) \qquad \forall n \in \mathbb{N} , \\ &k_n \in K(\boldsymbol{x}_n) \qquad \forall n \in \mathbb{N} , \end{aligned} \tag{3}$$

will have the property: $\exists x \in \mathbb{X} \lim_{n\to\infty} d(\underline{x}_n, x) = \lim_{n\to\infty} d(\overline{x}_n, x) = 0$, nor even the weaker one: $\lim_{n\to\infty} d(\underline{x}_n, \overline{x}_n) = 0$.

In fact, not all such sequences $(\boldsymbol{x}_n)$ are convergent to a point.

For example, if we bisect two-dimensional boxes over $\mathbb{R}$ (i.e. intervals over $\mathbb{R}^2$), bisecting only the first component, the subsequent boxes will become thinner and thinner, but will never converge to a point. (Nevertheless, it will converge to a two-dimensional box, with the first component thin.)

It is an interesting problem to consider – what (general) properties should the bisection operation have to be convergent to a point.

Theorem 3.1. *Consider a dense space $(\mathbb{X}, \preceq)$. The sequence (3) is convergent, at least to an interval (i.e. $\exists [\underline{x}, \overline{x}] \in \mathbb{IX}$ such that $\lim_{n\to\infty} d(\underline{x}_n, \underline{x}) = 0$ and $\lim_{n\to\infty} d(\overline{x}_n, \overline{x}) = 0$), if and only if $\mathbb{X}$ is a continuum.*

Proof. The (MO) axiom together with (1) imply that $(\underline{x}_n)_{n\in\mathbb{N}}$ and $(\overline{x}_n)_{n\in\mathbb{N}}$, obtained from (3) are both Cauchy sequences. Hence, they are convergent in any continuum.

Obviously $\forall n \in \mathbb{N}\ \underline{x}_n \preceq \overline{x}_n$, so $\lim_{n\to\infty} \underline{x}_n \preceq \lim_{n\to\infty} \overline{x}_n$, consequently the limit $\boldsymbol{x}$ of $(\boldsymbol{x}_n)$ is a proper interval.

On the other hand, in a non-continuum space, both $(\underline{x}_n)_{n\in\mathbb{N}}$ and $(\overline{x}_n)_{n\in\mathbb{N}}$ may converge to points not belonging to $\mathbb{X}$. (Example: $\mathbb{X} = \mathbb{Q}$ is the set of rational numbers; $\boldsymbol{x} = [1, 2]$; after each bisection we choose the interval, containing $\sqrt{2}$; the limit of both upper and lower bound sequences is $\sqrt{2} \notin \mathbb{X}$.) QED.

Remark 3.2. It seems that (3) converges to a point at least if all of the following take place:

- $\mathbb{X}$ is a continuum (see Theorem 3.1),
- all elements from $K(\cdot)$ appear in (k_n) infinitely many times (compare [1] for the theory of chaotic and semi-chaotic iterations),
- the set $K(\boldsymbol{x})$ "exhausts" all possibilities of bisecting $\boldsymbol{x} \in \mathbb{X}$.

But how to formalize the third assumption ... ?

Please note also the above conditions may be sufficient, but are not necessary. For example infiniteness of $K(\cdot)$ does not preclude convergence of the bisection process, as we shall see in Subsect. 5.2.

3.4 Problem two

Please note also that the following definition of bisection in $\mathbb{IR}$ fulfills the axioms of bisection:

$$b\big([\underline{x}, \overline{x}], k\big) = \Big([\underline{x}, \frac{1}{4} \cdot \underline{x} + \frac{3}{4} \cdot \overline{x}],\ [\frac{3}{4} \cdot \underline{x} + \frac{1}{4} \cdot \overline{x}, \overline{x}]\Big)\ ,\ k \in \{e\}\ .$$

This operation, for example, would "bisect" the interval $[0, 1]$ to $[0, 0.75]$ and $[0.25, 1]$, causing an unnecessary reproduction of a large part of the bisected interval.

The only solution, the author can think of, is to extend the structure from Section 2 by one more element: $\Big(\mathbb{X},\ d(\cdot, \cdot),\ \mu(\cdot),\ \preceq\Big)$, where $(\mathbb{X}, \sigma(\mathbb{IX}), \mu)$ is a measurable space; $\sigma(\mathbb{IX})$ is the smallest σ-field of subsets of $\mathbb{X}$, containing all intervals over $\mathbb{X}$.

Now, we can simply demand that:

$$\mu\big([\underline{m}, \overline{m}]\big) = 0\ . \tag{B-measure}$$

Example. For $\mathbb{IR}$ and $\mathbb{IR}^n$ the well-known Lebesgue measure seems a very good choice.

4 Operations on intervals

Suppose, we have an operation $\circ \colon \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X}$. Good examples are arithmetic operations on the elements of $\mathbb{R}$.

How to design the operation $\circ \colon \mathbb{IX} \times \mathbb{IX} \rightarrow \mathbb{IX}$, so that $x \in \boldsymbol{x}$ and $y \in \boldsymbol{y}$ would imply $x \circ y \in \boldsymbol{x} \circ \boldsymbol{y}$?

Obviously, $\boldsymbol{x} \circ \boldsymbol{y} = \square\{x \circ y \mid x \in \boldsymbol{x} \text{ and } y \in \boldsymbol{y}\}$ (where the interval hull $\square S$ of a set S is the smallest interval containing S) is a good choice. But is it always simple to be computed precisely ?

The general case may be complicated, but it is easy to prove that if "$\circ$" is monotonous, i.e. if $x_1 \preceq x_2$ and $y_1 \preceq y_2$ implies $x_1 \circ y_1 \preceq x_2 \circ y_2$, then $\boldsymbol{x} \circ \boldsymbol{y} = [\underline{x} \circ \underline{y}, \overline{x} \circ \overline{y}]$. A good example is the addition operation over $\mathbb{IR}$ or $\mathbb{IC}_{\text{rect}}$: it is given by $[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$. Subsection 5.1 gives us other good examples.

5 Intervals over other sets

5.1 Twins – intervals of intervals

There are a few types of twin arithmetic (e.g. [8]), but – to the best knowledge of the author – no bisection has been defined for them, up to now.

It is well-known that a metric may be defined on the set $\mathbb{IR}$ (or $\mathbb{KR}$):

$$d(\boldsymbol{x}, \boldsymbol{y}) = \max\left\{|\underline{x} - \underline{y}|,\, |\overline{x} - \overline{y}|\right\} . \tag{4}$$

Equation (4) may be found e.g. in [5], but – to the best knowledge of the author – it was proposed by Moore, already in his early works.

Remark 5.1. Please note, this is not the only metric that can be defined on the set $\mathbb{IR}$ – e.g. $d_1(\boldsymbol{x}, \boldsymbol{y}) = |\underline{x} - \underline{y}| + |\overline{x} - \overline{y}|$ or $d_2(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(\underline{x} - \underline{y})^2 + (\overline{x} - \overline{y})^2}$ would fulfill the axioms of metric, too.

At least three partial orders on $\mathbb{IR}$ have been introduced in the literature:

(i) the p.o. defined by inclusion relation; $\boldsymbol{x} \preceq \boldsymbol{y}$, iff $\boldsymbol{x} \subseteq \boldsymbol{y}$, i.e. iff $\underline{x} \geq \underline{y}$ and $\overline{x} \leq \overline{y}$, e.g. [4], [9],

(ii) the p.o. defined by relation "$\leq$", where $\boldsymbol{x} \leq \boldsymbol{y}$, iff $\forall x \in \boldsymbol{x}\, \forall y \in \boldsymbol{y} \quad x \leq y$, i.e. iff $\overline{x} \leq \underline{y}$, e.g. [6],

(iii) the p.o. defined by relation "$\leq_c$", where $\boldsymbol{x} \leq_c \boldsymbol{y}$, iff $\underline{x} \leq \underline{y}$ and $\overline{x} \leq \overline{y}$, e.g. [9],

First two orders have good set-theoretical interpretation, while the third one does not. Unfortunately, p.o. (ii) is not dense – no interval can be put between $[x_1, x_2]$ and $[x_2, x_3]$.

The first partial order, defined by the inclusion relation seems very good and can be used to define intervals over the set $\mathbb{IR}$ of intervals. Actually, so does the p.o. (iii), but the author cannot see any application for this ordering.

Now, let us focus on p.o. (i) and metric (4) and define the set $\mathbb{IIR}$ of intervals over $\mathbb{IR}$. We use the metric (4) and "$\subseteq$" as the p.o.

Elements $\Big[[\underline{x}, \overline{x}],\, [\underline{y}, \overline{y}]\Big]$ of $\mathbb{IIR}$ are sets of intervals $\boldsymbol{z}$ such that: $[\underline{x}, \overline{x}] \subseteq \boldsymbol{z} \subseteq [\underline{y}, \overline{y}]$. Obviously, the set is nonempty only when $[\underline{x}, \overline{x}] \subseteq [\underline{y}, \overline{y}]$.

We shall denote such intervals of intervals by boldface Gothic letters, e.g.:

$$\boldsymbol{\mathfrak{x}} = [\underline{\mathfrak{x}}, \overline{\mathfrak{x}}] = [\boldsymbol{x}_{in}, \boldsymbol{x}_{out}] = \Big[[\underline{x}_{in}, \overline{x}_{in}],\, [\underline{x}_{out}, \overline{x}_{out}]\Big]$$

So, $\boldsymbol{y} \in \boldsymbol{\mathfrak{x}}$, iff $\boldsymbol{x}_{in} \subseteq \boldsymbol{y} \subseteq \boldsymbol{x}_{out}$.

To make $\mathbb{IR}$ a measurable space, let us define a measure on this set. Consider any immersion of $\imath \colon \mathbb{IR} \to \mathbb{R}^2$, as in [10]. The standard immersion

$$\imath\Big(\big([\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], \ldots, [\underline{x}_n, \overline{x}_n]\big)^T\Big) = \big(-\underline{x}_1, -\underline{x}_2, \ldots, -\underline{x}_n; \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n\big)^T ,$$

seems to be a good choice.

Now, the measure of a set $Y \subseteq \mathbb{IR}$ of intervals is the Lebesgue measure of its image with respect to $\imath(\cdot)$:

$$\mu(Y) = \mu_L\Big(\{y \in \mathbb{R}^2 \mid y = \imath(\boldsymbol{x}) \text{ for some } \boldsymbol{x} \in Y\}\Big) , \qquad \forall\, Y \subseteq \mathbb{IR} . \tag{5}$$

It can be proved that (5) is a measure on the space $\mathbb{IR}$.

And now let us describe the bisection of intervals from $\mathbb{IIR}$.

Definition 5.2. Consider an interval of intervals $\mathfrak{x} = [\boldsymbol{x}_{in}, \boldsymbol{x}_{out}]$.

It can be bisected in two ways, i.e. $k \in K(\mathfrak{x}) = \{1, 2\}$.

$$b(\mathfrak{x}, 1) = \Big(\Big[[\underline{x}_{in}, \overline{x}_{in}], [\frac{\underline{x}_{in} + \underline{x}_{out}}{2}, \overline{x}_{out}]\Big], \ \Big[[\frac{\underline{x}_{in} + \underline{x}_{out}}{2}, \overline{x}_{in}], [\underline{x}_{out}, \overline{x}_{out}]\Big] \Big) \quad (6)$$

$$b(\mathfrak{x}, 2) = \Big(\Big[[\underline{x}_{in}, \frac{\overline{x}_{in} + \overline{x}_{out}}{2}], [\underline{x}_{out}, \overline{x}_{out}]\Big], \ \Big[[\underline{x}_{in}, \overline{x}_{in}], [\underline{x}_{out}, \frac{\overline{x}_{in} + \overline{x}_{out}}{2}]\Big] \Big) \quad (7)$$

The set $K(\boldsymbol{x}) \subseteq \{1, 2\}$ is determined by:

$$1 \in K(\boldsymbol{x}) \text{ , iff } \quad \underline{x}_{out} < \underline{x}_{in} \ ,$$
$$2 \in K(\boldsymbol{x}) \text{ , iff } \quad \overline{x}_{in} < \overline{x}_{out} \ .$$

Remark 5.3. Definition 5.2 is much simpler than it may look, at the first glance.

Actually, what does $\boldsymbol{x} \in \mathfrak{x}$ mean ? This means that $\underline{x} \in [\underline{x}_{out}, \underline{x}_{in}]$ and $\overline{x} \in [\overline{x}_{in}, \overline{x}_{out}]$.

We can bisect such an object in two ways: either ($k = 1$) divide the range of $\underline{x}$ to $[\underline{x}_{out}, \frac{\underline{x}_{in}+\underline{x}_{out}}{2}]$ and $[\frac{\underline{x}_{in}+\underline{x}_{out}}{2}, \underline{x}_{in}]$, or ($k = 2$) divide the range of $\overline{x}$ analogously.

This is literally, what equations (6) and (7) mean.

Property 5.1.1. Bisection of twins, as prescribed in Definition 5.2, fulfills axioms of bisection (B1)–(B3). It also fulfills – at least for measure (5) – axiom (B-measure).

5.2 Interval random variables – intervals of random variables

In recent years the author was working on the theory of so-called *interval random variables*, a notion related to p-bounds and Dempster–Shafer uncertainty – see e.g. [7] and references therein.

The title of this Subsection should probably be "generalized histograms – intervals of probability distributions", but in the author's opinion it would be less clear, though more accurate.

Definition 5.4. Let the probability space (Ω, S, P) be given, where Ω is the set of elementary events, S – the σ-field of events and P – the probability measure.

Any mapping $\boldsymbol{X} \colon \Omega \to \mathbb{I}_X \subseteq {}^*\mathbb{IR}$, for which sets $\{\omega \in \Omega \mid \boldsymbol{X}(\omega) = \boldsymbol{x} \text{ and } \boldsymbol{x} \in \mathbb{I}_{\boldsymbol{X}}\}$ are events (i.e. they belong to S), is called an *interval random variable*.

Definition 5.5. Consider a finite subset of $\mathbb{IR}$, $\mathbb{I}_{\boldsymbol{X}} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\}$.

A *generalized histogram* is a mapping $P \colon \mathbb{I}_{\boldsymbol{X}} \to \mathbb{R}_+ \cup \{0\}$, such that: $\sum_{i=1}^n P(\boldsymbol{x}_i) = 1$.

As in [7], we shall not always precisely distinguish between a random variable and its distribution.

To interpret histograms as intervals of probability distributions we need a partial ordering and a metric on the space of distributions.

We can use the following (simple and well-known) partial order "$\preceq$" on probability distributions:

$$X \preceq Y \text{ , iff } F_X(x) \le F_Y(x) \qquad \forall\, x \in \mathbb{R} \ , \quad (8)$$

where $F_X(\cdot)$ and $F_Y(\cdot)$ are cumulative distribution functions (CDFs) of random variables X and Y respectively (precisely: are two CDFs associated with two probability distributions; we do not consider random variables themselves, i.e. associating values to elementary events, we consider only their distributions).

We need a metric, yet. It may be the following function: $d(F_1, F_2) = \int\limits_{-\infty}^{+\infty} \big|F_1(x) - F_2(x)\big| dx$. Actually, it is an *extended metric*, i.e. it can attain infinite values for some pairs of arguments.

Now, an interval of random variables (i.e. an interval random variable) is the set $\boldsymbol{X} = [\underline{X}, \overline{X}] = \{X \mid \underline{X} \preceq X \preceq \overline{X}\}$, where $\underline{X}$ and $\overline{X}$ are real-valued random variables. As we said before, precisely: $\boldsymbol{X}$ is an interval of probability distributions and $\underline{X}$ and $\overline{X}$ represent real valued probability distributions.

How about bisection of such intervals ? A generalized histogram is a set of N pairs $(\boldsymbol{x}_i, p_i)$ – an interval and its probability. The number of bisection nodes is also N; please note it is not bounded in general – a histogram may consist of arbitrary many pairs. If the bisection is performed with respect to pair $\big([\underline{x}_i, \overline{x}_i], p_i\big)$, two histograms are obtained, each with $N + 1$ pairs. The bisected pair is replaced by either:

- $\big([\underline{x}_i, \mathrm{m}id\, \boldsymbol{x}_i], \frac{p_i}{2}\big)$ and $\big([\underline{x}_i, \overline{x}_i], \frac{p_i}{2}\big)$ or
- $\big([\mathrm{m}id\, \boldsymbol{x}_i, \overline{x}_i], \frac{p_i}{2}\big)$ and $\big([\underline{x}_i, \overline{x}_i], \frac{p_i}{2}\big)$.

This corresponds to two possibilities: the CDF either is larger than a specific value at the point $\mathrm{m}id\, \boldsymbol{x}_i$ or not. Since the CDF is non-decreasing there is no third possibility.

Example. Consider a histogram $\{(\boldsymbol{x}_i, p_i), i = 1, \ldots, N\}$, where $N = 2$, $\boldsymbol{x}_1 = [0, 2]$, $\boldsymbol{x}_2 = [2, 4]$ and $p_1 = p_2 = 0.5$.

We can bisect this histogram in two ways, obtaining either:

- $\Big\{\big([0, 1], 0.25\big), \big([0, 2], 0.25\big), \big([2, 4], 0.5\big)\Big\}$ and $\Big\{\big([1, 2], 0.25\big), \big([0, 2], 0.25\big), \big([2, 4], 0.5\big)\Big\}$
  or
- $\Big\{\big([0, 2], 0.5\big), \big([2, 3], 0.25\big), \big([2, 4], 0.25\big)\}$ and $\Big\{\big([0, 2], 0.5\big), \big([3, 4], 0.25\big), \big([2, 4], 0.25\big)\Big\}$.

Please note that bisection of a histogram, consisting of two intervals results with histograms of three intervals.

6 Applications

The presented theory not only allows us to explain known algorithms in a more general framework, but also to propose new ones. Two applications are briefly considered – one is conjuncted with twins and the other one – with interval random variables.

6.1 How to find an algebraic solution of a polynomial interval equation ?

Shary in [10] describes the problem of finding an *algebraic solution* of a linear equations system with interval coefficients, i.e. an interval (not necessarily proper) that, when substituted to the equation, results in a valid equality. For linear equations systems some algorithms to compute an algebraic solution are presented and the interpretation of such solutions is given.

It is uncertain if this interpretation holds for the nonlinear case or another one has to be constructed. According to [3] (Theorems 4.8 and 4.9) we should possibly change all but one occurrences of the variable to its duals.

Here is a simple example. We consider the equation $\boldsymbol{x}^3 - [1, 2] \cdot \boldsymbol{x}^2 + [-1, -2] = 0$. As the initial twin we take $\Big[[10, -11], [-11, 10]\Big]$.

We immediately obtain the following solutions: $\Big[[1.955065, 1.799000], [1.955064, 1.799000]\Big]$, $\Big[[1.955064, 1.799000], [1.955063, 1.799000]\Big]$, $\Big[[1.955064, 1.798999], [1.955063, 1.799000]\Big]$, $\Big[[1.955063, 1.798999], [1.955063, 1.799000]\Big]$. They probably form a cluster, enclosing one solution.

And if we solve $\boldsymbol{x}^3 - [1, 2] \cdot \mathrm{d}ual\, \boldsymbol{x}^2 + [-1, -2] = 0$, we obtain 8 points: $\Big[[2.205570, 1.695621], [2.205569, 1.695622]\Big]$, $\Big[[2.205570, 1.695621], [2.205569, 1.695621]\Big]$, $\Big[[2.205569, 1.695621], [2.205569, 1.695622]\Big]$, $\Big[[2.205569, 1.695621], [2.205569, 1.695621]\Big]$, $\Big[[2.205569, 1.695621], [2.205568, 1.695622]\Big]$, $\Big[[2.205569, 1.695621], [2.205568, 1.695621]\Big]$, $\Big[[2.205568, 1.695621], [2.205568, 1.695622]\Big]$, $\Big[[2.205568, 1.695621], [2.205568, 1.695621]\Big]$, that probably form a cluster, too.

Please note that in the inclusion function we have to use $\Big[\mathrm{d}ual\, \boldsymbol{x}_{out}, \mathrm{d}ual\, \boldsymbol{x}_{in}\Big] \neq \mathrm{d}ual \Big[\boldsymbol{x}_{in}, \boldsymbol{x}_{out}\Big]$.

6.2 How to find the maximal variance of all selections of an interval random variable ?

While there probably exists an efficient algorithm to compute the lower bound of the variance, the upper bound cannot be obtained efficiently. Nevertheless, using the proposed framework, we can compute it using the branch-and-bound meta algorithm, subsequently bisecting the histogram, representing the approximate probability distribution.

Also an analog of the monotonicity test was used to increase the efficiency. This approach may possibly be inefficient sometimes, but adding the monotonicity test resulted with a quite good performance.

For example computation for initially 4 discretization intervals and probability 0.25 for each of them obtained the value $[2.5, 2.5]$ for variance after 0.32 seconds and 1315 bisections (starting form range $[0.125, 4.25]$ for the variance). The resulting random variable was discrete with 10 points. Hence for probabilities $p_1 = p_4 = 0.1$, $p_2 = p_3 = 0.4$ the program computed variance as $[1.599999, 1.600001]$ (when the initial approximation: $[0.049999, 3.050001]$) in 0.01 seconds and 51 bisections. As previously, the resulting random variable was discrete with 10 points. In both experiments $\boldsymbol{x}_i = [i - 1, i]$, where $i = 1, \ldots, 4$, were used.

Obviously, computing the variance is quite an easy example, but it seems likely that similar techniques can be used to solve optimization problems over stochastic spaces; other interval techniques, like constraint propagation (e.g. [4]) may also be useful for intervals of probability distributions.

6.3 Possible extensions

Only the basic branch-and-bound approach was used in the calculations with twins and in random variables calculations also an analog of the monotonicity test. It seems an analog of the monotonicity test exists also for twins. And – probably – some interval Newton operators for both cases can also be obtained.

7 Conclusions

The paper described a trial to develop a generalized theory of interval computations, in particular it gave axioms of bisection and Theorem 3.1 on its convergence.

Both applications presented in Section 6 prove it is possible to do computations on intervals over sets of non-numbers. Some extensions – mentioned e.g. in Subsection 6.3 – are possible and may be the subject of future research. Also, some theoretical questions need to be answered, like what conditions assure the existence of points $\underline{m}$ and $\overline{m}$ for bisection or about the existence and usefulness of intervals over spaces where sets (2) are not open.

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