Identification of Cyclic and Chaotic Behavior in a Dynamical System Generated by Phenotypic Evolution

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Abstract. A discrete deterministic dynamical system generated by the expected value derived from the model of phenotypic evolution is considered. Depending on fitness functions and a standard deviation of mutation, the system converges not only to stable fixed points but also displays cyclic and chaotic behavior. To detect the phenomena an auto-correlation function, a phase space portrait and a power spectrum of trajectories of the system were exploited.

1 Introduction

Because evolutionary methods are indeterministic and complex, their theoretical examination usually relies on simplifying assumptions. Mathematical theory of dynamical systems is applied to analyze either infinite [9] or very small populations [2, 3, 4, 5]. The models of evolution allow the researcher to cope with stochastic noise shadowing some principal mechanisms in evolutionary processes.

The dynamical system model of phenotypic evolution presented in the paper is generated by expected values of states of two-element populations. This model of evolution is ruled by proportional selection and normally distributed mutation. The asymptotic analysis of the system displayed a rich spectrum of behavior for various fitness functions and for the parameter of the evolutionary process (the standard deviation of mutation σ). For the majority of fitness functions, the system converged to stable fixed points for small values of the standard deviation of mutation. Increasing σ, some fixed points disappeared and others become unstable. When the fixed point lost its stability, a pitchfork bifurcation appeared, having given rise to the orbit of period 2. As the standard deviation of mutation was further increased, the period doubling bifurcations and chaos were observed for specific functions (asymmetrical, with large basins of attraction of non-global optima). In the past, we identified periodic orbits and chaos observing trajectories of the dynamical system [3, 4]. It is not a very convenient or accurate method. In this paper cyclic and chaotic behavior are detected with the use of techniques taken from dynamical systems' theory: phase portraits, power spectra and auto-correlation functions. These techniques allow long-period orbits to be distinguished from chaos.

2 Dynamical System Model of Two-Individual Population

A special instance of phenotypic asexual evolution [1] is considered where the population of two evolves in one-dimensional continuous search space $P = \{x_1, x_2\}$. The reproduction is initialized with proportional selection followed by normally distributed mutation. The only parameter of the model is the standard deviation of mutation σ. The evolution is regarded in a space of possible populations called *the space of population states* S [2, 3, 4, 5]. For two-element populations, the space S is two dimensional and therefore it is easy to visualize. The population state should not depend on arrangement of individuals within the population. Thus, an equivalence relation, which makes states independent on permutations of individuals in a population, has to be defined. Consequently S becomes a factor space with equivalence relation $U : S = R^2/U$. For a population of two, the equivalence relation sorts individuals in an increasing order and the state space is identified with the half-plane bounded by the line (called *the identity axis*) $x_1 = x_2$: $S = \{(x_1, x_2) : x_1 \geq x_2\}$. A population in the i-th generation is described by a state $\boldsymbol{s}^i = (x_1^i, x_2^i)$. It is more convenient to observe evolution in the space S by rotating counterclockwise the coordinate frame with the angle of $\pi/4$ around its origin. In the new frame, coordinates (w, z) become $w = (x_1 - x_2)/\sqrt{2}$, $z = (x_1 + x_2)/\sqrt{2}$. Now, the state space occupies the right half-plane ($w \geq 0$) bounded by the Z-axis and the population in the i-th generation is described by the state $\boldsymbol{s}^i = (w^i, z^i)$. The new coordinate w is a measure of population's diversity and describes a distance of the population state from the identity axis, where both individuals have the same trait. Coordinate z situates a state along the identity axis and is related to the mean of population P.

The somewhat complicated structure of the factor space S in the WZ frame may cause some difficulties when describing the model but it allows knowledge to be gained concerning evolution. In the space of states, the expected value of the population state in the next generation can be calculated analytically. Expected values of coordinates w and z in the $(i+1)$-st generation are given by the equations

$$\begin{cases} E_{i+1}[w|\boldsymbol{s}^i] = & \sqrt{\frac{2}{\pi}}\sigma + (1 - {\Psi^i}^2) \cdot \sigma \cdot \Theta(w^i/\sigma) \\ E_{i+1}[z|\boldsymbol{s}^i] = & z^i + \Psi^i \cdot w^i, \end{cases} \tag{1}$$

where $q(x)$ denotes a fitness of the individual x and

$$q_1 = q(x_1) = q((w+z)/\sqrt{2}), \quad q_2 = q(x_2) = q((z-w)/\sqrt{2}), \quad \Psi(w, z) = \frac{q_1 - q_2}{q_1 + q_2},$$

$$\Psi^i = \Psi(w^i, z^i), \quad \Theta(\xi) = \phi_0(\xi) + \xi\Phi_0(\xi), \quad \phi_0(\xi) = \frac{1}{\sqrt{2\pi}}(\exp\left(-\frac{\xi^2}{2}\right) - 1),$$

$$\Phi_0(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\xi \exp(-\frac{t^2}{2})dt.$$

Expected values generate a two dimensional discrete dynamical system

$$(w, z) \longrightarrow F(w, z) = \begin{bmatrix} F_1(w, z) = E_{i+1}[w|\boldsymbol{s}^i] \\ F_2(w, z) = E_{i+1}[z|\boldsymbol{s}^i] \end{bmatrix}. \tag{2}$$

The asymptotic behavior of dynamical system (2) was studied and fixed points and their stability were determined. Fixed points (w^s, z^s) of the system are characterized by equations

$$w^s \simeq 0.97\sigma, \tag{3}$$

$$\Psi(w^s, z^s) = 0. \tag{4}$$

The w-coordinate of fixed points (3) depends on the standard deviation of mutation σ and it does not depend on the fitness function q. The z-coordinate depends on the fitness function as the condition $q((z^s+w^s)/\sqrt{2}) = q((z^s-w^s)/\sqrt{2})$ must be satisfied. Modality and shape of the fitness function determine the number of fixed points. For uni-modal fitness functions the dynamical system has one fixed point whereas for multimodal fitness with k optima the system has got no more than $2k+1$ fixed points. Fixed points are located in the neighborhood of optima and in the saddles of the function. The value of standard deviation of mutation σ impacts the number of fixed points. Their number decreases with an increased value of σ. When fitness functions are symmetrical, fixed points are located on the symmetry axis. Asymmetry in a fitness function influences the z-coordinate of the fixed point.

Stability of fixed points is determined by eigenvalues of the linear approximation (the Jacobi matrix) of the system (2) at the points. For dynamical system (2) the matrix is diagonal and its eigenvalues are equal to $\lambda_1 = \Phi_0(w^s)$, $\lambda_2 = 1 + w^s \frac{\partial \Psi(w,z)}{\partial z}|_{(w^s,z^s)}$. As $|\lambda_1| < 1$, a fixed point stability depends on the second eigenvalue only. λ_2 depends on the fitness and on the standard deviation of the mutation. Fixed points in a vicinity of saddles are always unstable. Optima fixed points are usually stable for small values of the standard deviation of mutation. When the value of σ is increased, stability can be lost and periodic orbit and chaos can be observed. Chaos emerged when the fitness function was asymmetrical or the attraction regions for some optima were very weak.

Two kinds of fitness functions are used to show cyclic and chaotic behavior of dynamical system (2): the asymmetric version of the bell-shape uni-modal Gaussian function with branches of different slopes a_i $(i = 1, 2)$

$$q(x) = \begin{cases} \exp(-a_1 x^2) & \text{for } x \leq 0 \\ \exp(-a_2 x^2) & \text{for } x > 0 \end{cases} \tag{5}$$

and the sum of two Gaussian functions exemplifying bimodal fitness

$$q(x) = \exp(-a_1 x^2) + h \cdot \exp(-a_2(x-1)^2). \tag{6}$$

Varying values of parameters of function (6) (peaks slopes decline a_1, a_2, the second hill height h) adaptive landscapes with different characteristics (symmetrical, asymmetrical, various areas of attraction) can be examined [6].

Trajectories of dynamical system (2) converge to a fixed point in the landscape of uni-modal asymmetrical fitness (5) (Fig. 1a). The fixed point is stable for small values of the standard deviation of mutation until $\lambda_2 > -1$. When $\lambda_2 = -1$, the critical value of standard deviation of mutation is calculated $\sigma_c = \sqrt{2/\sqrt{a_1 \cdot a_2}}/0.97$. For $\sigma{=}\sigma_c$ the pitchfork bifurcation appeared. When σ exceeds its critical value, the fixed point became unstable and the orbit of period 2 arises. Further increasing the standard deviation of mutation, a series of period-doubling bifurcations with periods of $4, 8, 16, \ldots$ can be observed.

For bimodal fitness function (6), limit points of trajectories are determined by initial states (more precisely, to which optimum (local or global) basin of attraction the points

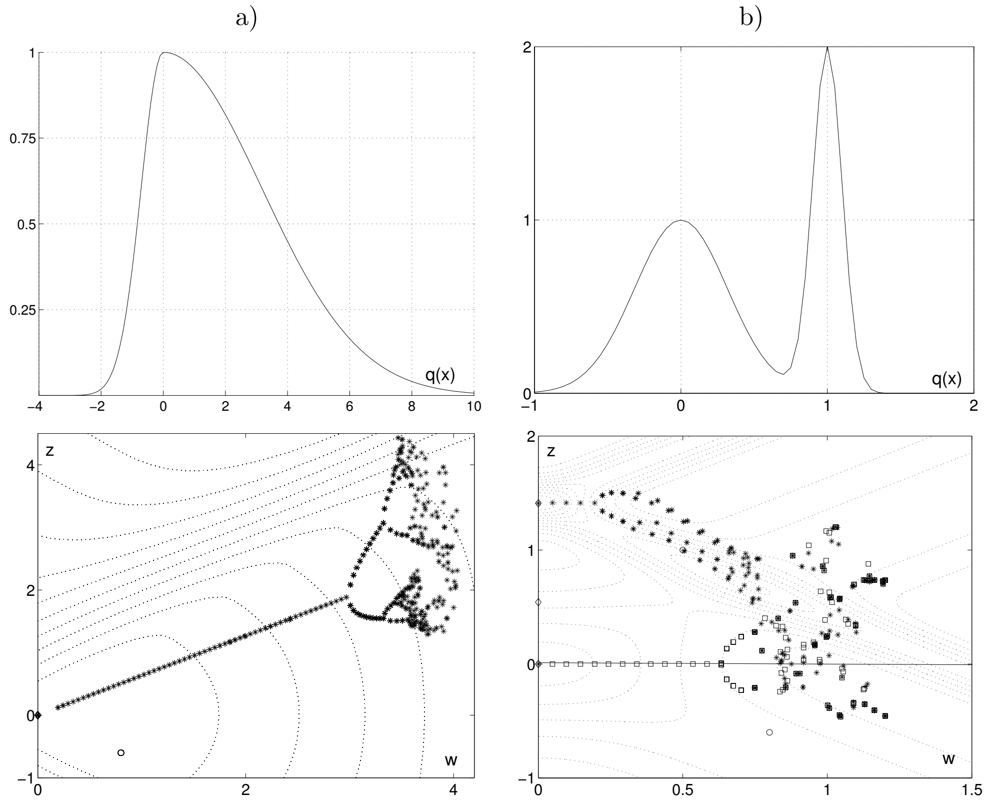

Figure 1. Fitness functions profiles (upper) and equilibrium points of trajectories of expected values (down). a) uni-modal asymmetrical fitness function (5), ($a_1 = 1, a_2 = 0.05$), initial state $s_0 = (0.8, -0.6)$ b) bimodal fitness function (6), ($h = 2, a_1 = 5, a_2 = 50$), initial states $s_0^1 = (0.8, -0.6)$ and $s_0^2 = (0.5, 1.0)$. Initial states are marked with circles while equilibrium points with asterisks (a) and squares (for s_0^1) and asterisk (for s_0^2) (b).

belong, cf. Fig. 1b). Because the basin of attraction of the global optimum is weak ($a_1 \ll a_2$), with increasing σ, orbits of both hills tend to each other and, finally become inseparable. Period-doubling bifurcations and chaos are also observed in this case.

3 Detecting Periodical Orbits and Chaos

Bifurcations and chaos were identified in numerical experiments by observing time series and their equilibrium points (Fig. 1). It is the simplest but usually not very accurate method to deal with those phenomena. In dynamical system theory a few techniques for detection chaos and/or periodical orbits from time series are used [7, 8]. *A phase space portrait* presents dependence of velocities along the trajectory as a function of their anchoring positions. The portrait can be plotted by recording two system variables directly (x, y), the variable with its velocity $(x, \dot{x})$ or the delayed variable $(x(t), x(t-\tau))$, where τ is the delay time. Similarly to monitoring time series, it is not a very precise method. There are non-chaotic systems for which phase trajectories look similar to those with chaos. Calculating *a power spectrum* of a signal (the trajectory observed over a given time horizon) is difficult and usually demands using numerical recipes (DFT or FFT) but this technique is more reliable. The spectrum is given by the squared norm of the Fourier transform $X(\omega)$ of the time series $x(t) : P(\omega) = |X(\omega)|^2$. For chaotic systems there are many frequencies and energy is concentrated in lower range of frequencies. Another detection method, *an auto-correlation function* is equivalent to Fourier transform of a power spectrum. For time series $x(t)$ with expected value $E[x(t)]$ the auto-correlation is defined as $C(\tau) = E[x(t)x(t+\tau)] - E[x(t)]^2$. The function depends on the shift parameter τ between trajectory and its translated copy. It is a measure of self-similarity in trajectories. Processes which do not display unstable behavior are strongly correlated and their auto-correlation functions are constant or oscillate. For chaotic systems the function is narrow and approaches zero very fast.

When implementing the auto-correlation function it is advised to keep a large amount of data to make it less sensitive on averaging over short time intervals (shifted signal should fall into the same domain as the original). To compute the power spectrum of the signal the number of data should equal to an integer power of 2. In this case the most effective butterfly algorithm can be applied to compute the Fourier transform. Although processing massive data can be quite time consuming to detect orbits with long periods they may be necessary. For a slow process (for example when the mutation rate is very small) data to process by the auto-correlation function and the power spectrum technique should be gathered omitting registration of first few hundred iterations to avoid transient states of the process.

For dynamical system (2) bifurcations and chaos observed in time series are discovered using aforementioned methods. In Figs.2-4 simulation results are presented for uni-modal asymmetrical Gaussian function (5) and bimodal Gaussian function (6) and a few values of the standard deviation of mutation. In all simulations the more informative z coordinate is examined. For the bimodal fitness two initial population states were evaluated: the state $s_0 = (0.8, -0.6)$ belongs to the basin of attraction of the local optimum while $s_0 = (0.5, 1.0)$ is attracted by the global optimum. Values of the standard deviation of mutation were selected based on equilibrium points in Fig. 1 where periodical orbits and chaos were observed. For uni-modal asymmetrical fitness function orbits of period $2, 4$ and 8 were noticed ($\sigma = 3.2, 3.7, 3.85$ respectively). All three methods confirmed these observations. More detailed bifurcation diagrams indicated chaos for $\sigma = 4.0$. Although the phase portrait looks more chaotic than the others, both power spectrum and auto-correlation showed that it is probably an orbit of large period.

Confirmation that the dynamical system (2) behaves chaotically for some standard deviation of mutation can be found in Fig. 3, 4. The phase portrait looks chaotic, the signal power is spread for all frequencies (although two of them are distinguishable) and the auto-correlation function decreases to zero for $\sigma = 0.9$ and the initial state is placed in the basin of attraction of the local optimum. Similarly, the methods imply chaos for $\sigma = 0.8$ and for the other initial states. The last phase diagram in Fig. 3 shows that for the standard deviation of mutation close to 1.2 the orbit of period 3 is encountered, that, according to the Sarkovsky theorem, indicates orbits of all other periods.

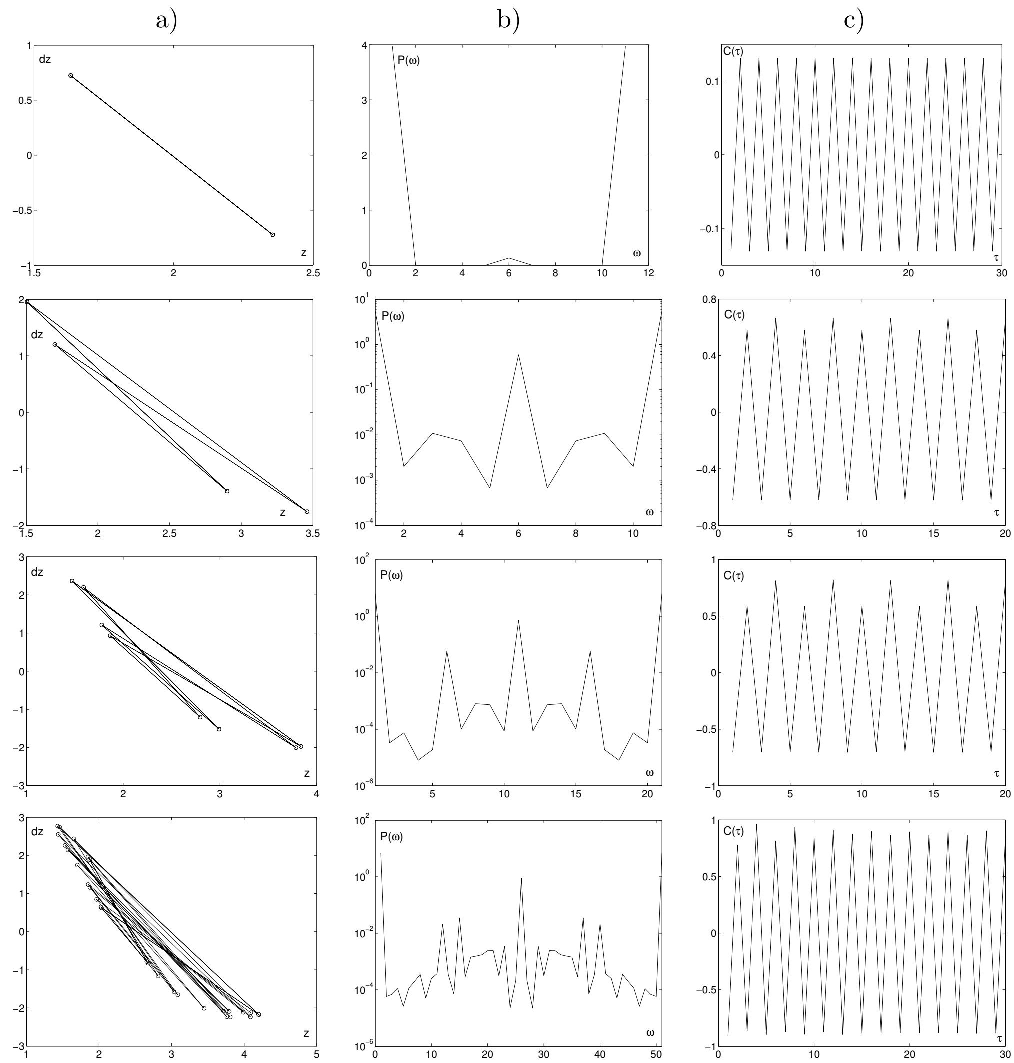

Figure 2. Phase portrait (a), power spectrum (b), autocorrelation function (c) for uni-modal asymmetrical fitness function (5), $a_1 = 1$, $a_2 = 0.05$, $\sigma = 3.2, 3.7, 3.85, 4.0$ (from top to bottom)

4 Conclusions

In this paper the study of unstable behavior of the dynamical system generated by phenotypic evolution is presented. The focus is on detecting periodical orbits and chaos by using techniques from dynamical systems theory. Methods of phase portraits, power spectrum and auto-correlation function were exploited. Two kinds of fitness functions were examined: uni-modal asymmetrical and bimodal with distinct basins of attraction of both hills, for which the periodical orbits and series of period doubling bifurcations

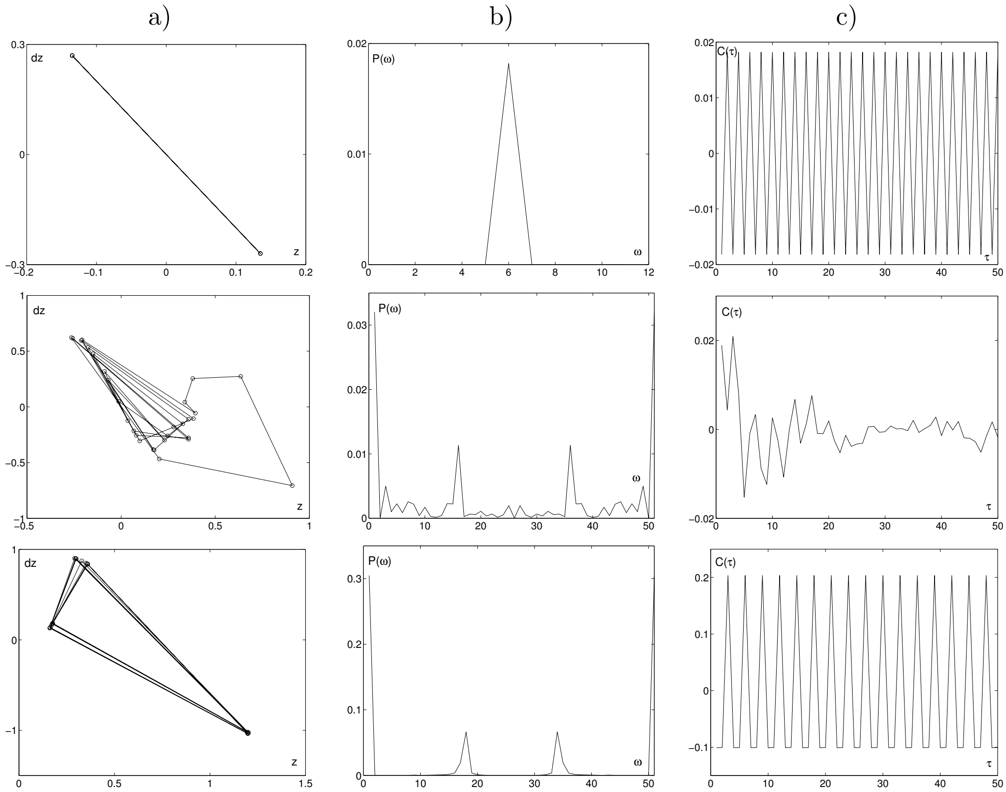

Figure 3. Phase portrait (a), power spectrum (b), auto-correlation function (c) for bimodal fitness function (6), ($h = 2, a_1 = 5, a_2 = 50$), initial state $s_0 = (0.8, -0.6)$, $\sigma = 0.7, 0.9, 1.2$, (from top to bottom)

leading to chaos were observed. The analysis confirmed orbits of different periods and chaos. Using analytic techniques for detecting chaos seems to be reasonable, because relying on evaluation of time series only can easily cause misclassifications. Sometimes, as was shown, the expected chaos appeared to be an orbit of a long period. The regarded methods are relatively fast and easy to implement but results are sometimes questionable and depend on parameter settings. The more reliable but computationally more expensive method of detecting chaos is based on Lyapunov exponents [5]. It is recommened to use a mix of all of the methods to increase the chance of drawing reliable conclusions.

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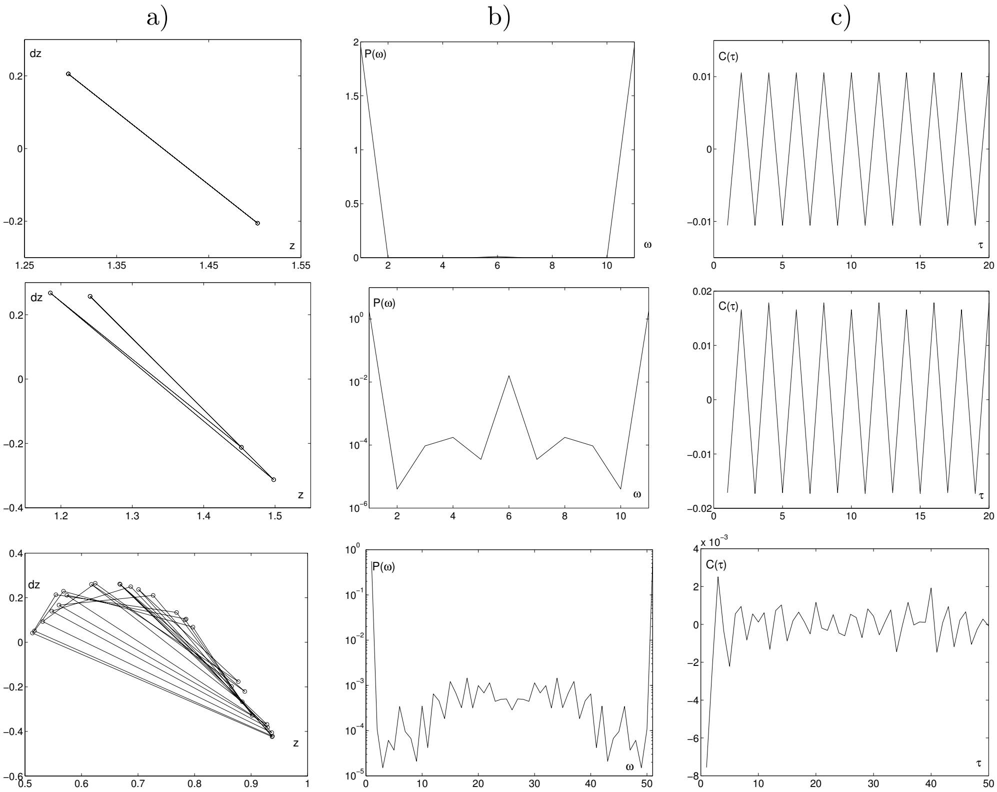

Figure 4. Phase portrait (a), power spectrum (b), auto-correlation function (c) for bimodal fitness function (6), $(h = 2, a_1 = 5, a_2 = 50)$, initial state $s_0 = (0.5, 1.0)$, $\sigma = 0.3, 0.4, 0.8$, (from top to bottom)

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